Robotic Manipulators and the Geometry of Real Semialgebraic Sets

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Abstract—**Some modern techniques are applied from real semialgebraic geometry to the robotic manipulator problem. In particular, using the notion of "metric entropy," the complexity of maneuvering the manipulator from state to state is discussed.**

Introduction

ONE OF THE key problems of robotics is the manipulation of rigid bodies by manipulators which are motor-driven kinematic chains. The purpose of this paper is to apply some modern techniques from real semialgebraic geometry to this central problem area.

The application of semialgebraic geometry to certain areas of robotics is by now standard. For example, consider the "piano movers' problem" discussed in [14] and [15]. Here one is interested in finding a continuous motion which will take one body (a "robot" or "robotic manipulator") from a given initial position to a desired final position. Of course, there are some geometric constraints (e.g., one does not want the robot colliding with a wall!). Under certain natural assumptions (see [14], [15], and Section I, to follow), one can show that the solution of the piano movers' problem amounts to finding an effective algorithm for computing when two points lie in the same connected component of a certain real semialgebraic set. In a sense, this problem has been solved by Tarski [19], and present work centers on finding more efficient algorithms for computing the solution (which are suitable for implementation on a digital computer).

In this paper we would like to apply certain recent results (e.g., [5]–[7], [22]–[24]) in the geometry of semialgebraic sets to the robotic manipulator problem (see Section I for a precise statement). We feel that a contribution of this work is the use of new techniques in the theory of *semialgebraic mappings* (see, e.g., [5], [22]–[24]) in studying robotic manipulators. Previous work has centered mainly on the static properties of real semialgebraic sets without considering algebraic morphisms between them. We will see, for example, that these ideas lead to a notion of "bad positions" for the manipulator defined in terms of the critical values of a certain map. When the manipulator is in such a state, one has little control over its position (see Section V for a precise definition). We will also be able to study the notion of complexity (precisely defined through "metric entropy") as applied to maneuvering the robot from state to state.

We should emphasize that all the algorithms we will give in this paper are effective. However, "effective" is certainly not synonymous with "practical," and at this point the practicality of implementing our procedures on a digital computer must be considered open.

Finally, many of the technical results on real semialgebraic sets we use here can be found in [22]. A good introduction to topology and geometry from a system theoretic point of view can be found in [18].

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I. Generalities on Robotic Manipulators

In this section we give a model of the type of robotic manipulator we will consider in this paper and discuss how the notion of the real semialgebraic set enters into its control. For related discussions see Brockett [2] and Paul [13].

The robotic manipulator we wish to consider consists of n rigid bodies $P_1, \cdots, P_n$ joined in a chain with joints at $x_1, \cdots, x_n$. The joint x_1 is rigidly placed at the origin of $\mathbb{R}^3$, and the last body P_n is identified with a "tooling device" P (i.e., we set $P = P_n$). We also fix some point x_{n+1} in P. This situation is illustrated in Fig. 1.

At each joint x_i $(i = 1, \cdots, n)$ we assume three degrees of freedom: the direction of the vector $\overrightarrow{x_ix_{i+1}}$ and the rotation angle of P_i around the axis $\overrightarrow{x_ix_{i+1}}$ (Fig. 2). Mathematically, this situation can be described as follows. We assign to each P_i an orthonormal frame $w_i = (e_1^i, e_2^i, e_3^i)$ at the point x_i $(i = 1, \cdots, n)$, where e_1^i is parallel to $\overrightarrow{x_ix_{i+1}}$ for $i \geqslant 1$, and let w_0 be the standard orthonormal frame in $\mathbb{R}^3$ at the origin. Clearly, the position of P_i with respect to P_{i-1} is completely described by the orthogonal transformation g_i of $\mathbb{R}^3$ (i.e., $g_i \in SO(3)$) which sends w_{i-1} to w_i $(i = 1, \cdots, n)$.

Now we can consider the g_i $(i = 1, \cdots, n)$ as the control parameters of the manipulator. Certainly, this choice is convenient mathematically and also constructively reasonable if the driving mechanisms are placed at the joints $x_1, \cdots, x_n$.

Set $r_i := \|x_{i+1} - x_i\|$, $i = 1, \cdots, n-1$ (where $\| \ \|$ denotes ordinary Euclidean distance). For given control

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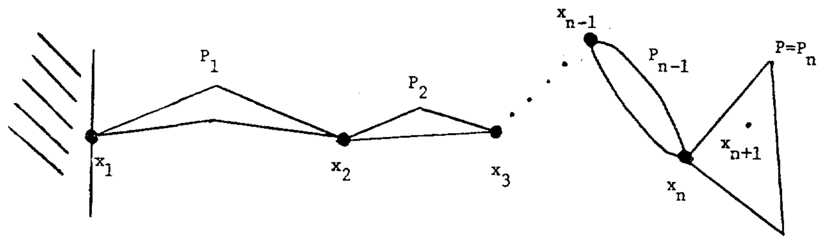

Fig. 1. Robotic manipulator.

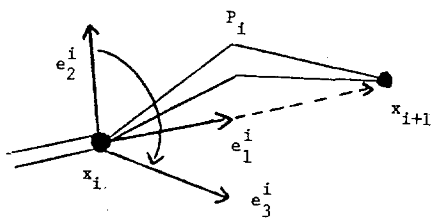

Fig. 2. Joint with three degrees of freedom.

parameters $g_1, \cdots, g_n \in SO(3)$, we have

$$\begin{aligned}
w_1 &= g_1(w_0)\\
x_2 &= r_1 g_1(e_1^0)\\
w_2 &= g_2 g_1(w_0)\\
x_3 &= x_2 + r_2 g_2 g_1(e_1^0) = (r_2 g_2 g_1 + r_1 g_1)(e_1^0)\\
&\vdots\\
w_{n-1} &= g_{n-1} g_{n-2} \cdots g_1(w_0)\\
x_n &= (r_{n-1} g_{n-1} \cdots g_1 + \cdots + r_2 g_2 g_1 + r_1 g_1)(e_1^0)\\
w_n &= g_n \cdots g_1(w_0).
\end{aligned} \tag{1}$$

(Note that the point x_{n+1} is irrelevant in describing the position of the tooling device P, and so we omit its expression from (1).) Equations (1) imply that the behavior of the robotic manipulator represented in Fig. 1 may be described by the algebraic mapping

$$\emptyset : A \to \mathbb{R}^3 \times SO(3)$$

where $A := SO(3) \times \cdots \times SO(3)$ (the Cartesian product is taken n times), and $\emptyset(g_1, \cdots, g_n) := (x_n, w_n)$.

It is well-known that $SO(3)$ is an algebraic three-dimensional submanifold of $\mathbb{R}^9$, and hence A is a $3n$-dimensional algebraic manifold. Note that the equalities (1) only involve matrix multiplication and addition. Consequently, $\emptyset$ is a polynomial mapping of degree n.

Now in any practical problem the situation is, of course, more complicated. Indeed, clearly, the admissible values of the control parameters g_i will be restricted by the construction of the manipulator and certainly by the (minimal) requirement of avoiding self-collisions. However, if the restrictions on the parameters are given by polynomial equations and inequalities and the bodies P_i $(i = 1, \cdots, n)$ are constructed from parts defined by polynomial inequalities and equations, then, arguing precisely as in [15], it is easy to show that the subset $A' \subset A$ of *admissible parameters* will also be defined in A by polynomial equations and inequalities. A' will also be referred to as the *control space of parameters*.

Thus we conclude that under natural assumptions the control of the robotic manipulator described in Fig. 1 is represented by the polynomial mapping $\emptyset := A' \to \mathbb{R}^3 \times SO(3)$. Clearly, then, the problem of control of the manipulator will be intimately related to the geometric properties of A' and the structure of $\emptyset$.

II. Generalities on Semialgebraic Sets

In this section we formally define the basic mathematical object (namely, "semialgebraic set") we will use in this paper, as well as discuss some computational properties of these objects as related to robotics. Thus we begin with the following definitions.

A. Definitions

1) A *semialgebraic set* A in $\mathbb{R}^n$ is a set which can be represented as a result of finitely many set-theoretic operations over sets of the form $\{f_i = 0\}$, $\{f_j > 0\}$ for real polynomials f_i, f_j.

2) A set-theoretic formula of this representation, together with the dimension n and the degrees of the polynomials f_i, f_j, is called the *diagram* $D(A)$ of the representation of A. If we do not specify a representation, then by slight abuse of notation $D(A)$ will denote the "diagram of some representation of A."

B. Remarks

1) Note that if we claim that some property of A depends only on the diagram $D(A)$, this means in particular that the explicit coefficients of polynomials defining A are not relevant, but that for the given property we are only interested in specifying the combinational data given by the number of polynomials and their degrees.

2) We will see later (in Section IV) that a key question in the robotic manipulator problem (and the piano movers' problem) is determining the maximal number of connected components of all the semialgebraic sets with a given diagram. For the algorithm discussed in [15] this number is bounded by a quantity which is *polynomial* in the number of polynomials and in their degrees and *exponential* in the number of degrees of freedom n.

We would like to show here that this is not an accidental result related to a specific algorithm, but instead a fundamental property in the computation of the betti numbers of a given

Fig. 3. Manipulator in bad position.

semialgebraic set. The fact that one gets quantities exponential in the number of degrees of freedom could of course have serious implications for the digital implementation of procedures involving the control of robotic manipulators.

Explicitly, let us see how to estimate the maximal number of connected components of the semialgebraic set $A := \mathbb{R}^2 \setminus \bigcup_{i=1}^m \{f_i = 0\}$ where the curves $\{f_i = 0\}$ are of degree d and in general position (see [18]). The arguments involved in the general case are similar (see [1], [21]). We will need the following two standard results (see [1] or [16]).

a) The Harnack Inequality: Let $f(x, y)$ be a polynomial of degree d. Then the number of connected components of the curve $\{f = 0\}$ in $\mathbb{R}^2$ is at most $(d^2 - d + 2)/2$.

b) The Bezout Theorem ([8] or [16]): Let $f_1(x, y)$, $f_2(x, y)$ be polynomials of degree d. Then the number of points of intersection of the curves $\{f_1 = 0\}$ and $\{f_2 = 0\}$ in $\mathbb{R}^2$ is at most d^2.

With a) and b) it is easy to bound now the number of connected components of A. Indeed, set $\hat{A} := \bigcup_{i=1}^m Y_i$, where $Y_i := \{f_i = 0\}$, $i = 1, \cdots, m$. An *edge* of $\hat{A}$ is then defined to be any smooth connected component of $\hat{A}$ or any segment in $\hat{A}$ joining two points of intersection of Y_i and Y_j ($i \neq j$).

Clearly,

$$\text{the number of connected components of } A \leqslant 2 \text{ the number of (edges of } A') \tag{2}$$

since each component is bounded by an edge, and each edge is counted twice.

Let μ_i be the number of components of Y_i, $i = 1, \cdots, m$, and let the number of intersection points of Y_i and Y_j, $i < j$ be denoted by d_{ij}. Then

$$\text{the number of edges} \leqslant \sum_{i=1}^m u_i + 2 \sum_{i<j} d_{ij}. \tag{3}$$

Let C be the number of connected components of A. Then from (2) and (3) (and using a) and b) from before) we have

$$C \leqslant 2m \left[\frac{d^2 - d + 2}{2}\right] + 4\,\frac{m(m-1)}{2}\,d^2$$
$$= (2m^2 - m)d^2 - md + 2m, \tag{4}$$

an expression which is polynomial in m and d and exponential in the number of degrees of freedom $n = 2$. Since in the derivation of (4) we used only basic theorems of geometry, it is clear that the fact that one gets a bound which is exponential in the number of degrees of freedom is a fundamental fact and not an accident connected to some method of computation [15].

Finally, we note that it is easy to write down examples of semialgebraic sets in which the number of connected components C is *equal to* an expression which is polynomial in the number of defining polynomials and their degrees and exponential in the number of degrees of freedom (as we derived in (4)). Indeed, let Y be the union of p lines in "general position" in $\mathbb{R}^2$. Then, clearly, if $A := \mathbb{R}^2 \setminus Y$, the number of connected components of A is precisely

$$C := \frac{p(p+1)}{2} + 1.$$

However, for $p = md$ we can interpret Y as the union of m curves of degree d (each of the m curves being a union of d straight lines). Hence

$$C = \frac{md(md+1)}{2} + 1$$
$$= \frac{m^2d^2 + md + 2}{2}$$

which is polynomial in the number of curves and their degrees and exponential in the number of degrees of freedom ($=2$). Thus, generally, the best possible upper bound for the number of connected components of a given real semialgebraic set will depend exponentially on the minimal dimension of a Euclidean space into which the set may be embedded, i.e., on the number of degrees of freedom.

III. The Set of Accessible Positions

In this section we would like to describe the set of positions of the tooling device P which can be attained for admissible values of the control parameters. This important problem was studied, e.g., in [4], [17], and [10]. Clearly (using the notation of Section I), the proper notion in this context should be as follows.

Definition: The space of *accessible positions* of the tooling device P is the image $\mathcal{O}(A') \subset \mathbb{R}^3 \times SO(3)$.

Now using standard and easy results from the theory of semialgebraic sets [3], one has that $\mathcal{O}(A')$ is a semialgebraic subset of $\mathbb{R}^3 \times SO(3)$. Moreover, the diagram of $\mathcal{O}(A')$ is determined by the diagram of A', and the polynomial equations and inequalities defining $\mathcal{O}(A')$ can be effectively found. Indeed, these remarks are simple corollaries of the Seidenberg–Tarski theorem [19], [3]. Explicit algorithms for such constructions based on a certain version of the Tarski "decision algorithm" are given in [15].

IV. On the Piano Movers' Problem

In this section we would like to precisely formulate the piano movers' problem and apply some of our geometric techniques to its solution to sharpen certain results from [14] and [15]. This problem is very similar to the robotic manipulator problem formulated in Section I, and, basically, the same ideas apply equally to the solution of both problems.

First recall [14], [15] that the piano movers' problem in $\mathbb{R}^3$ may be stated as follows. Given a body α (three-dimensional, bounded, and semialgebraic) and a three-dimensional open region bounded by a collection of semialgebraic surfaces ("walls"), find a continuous motion connecting two given positions and orientations of α such that α does not collide with any of the walls, or show that no such motion exists.

Exactly the same problem can be posed for the robotic manipulator. Indeed, as before let $\mathcal{O}(A') \subset \mathbb{R}^3 \times SO(3)$ denote the set of positions of the tooling device avoiding collisions (plus some other constraints; see Section I). Then, clearly, the piano movers' problem taken in this context amounts to finding whether two given positions (the initial and final positions) belong to the same connected component of $\mathcal{O}(A')$.

As we have seen, under certain natural assumptions $\mathcal{O}(A')$ is a semialgebraic set, and in [15] an algorithm is given for determining the connected components of $\mathcal{O}(A')$. Actually, using standard techniques from semialgebraic geometry, it is easy to prove the following theorem.

Theorem 1: With the foregoing notation, given points q_1, $q_2 \in \mathcal{O}(A')$ which belong to the same connected component of $\mathcal{O}(A')$, a semialgebraic curve s exists connecting q_1 and q_2 such that the diagram $D(s)$ is determined only by the diagram $D(\mathcal{O}(A'))$. Moreover, an effective procedure exists for determining the equations and inequalities defining s.

Proof: See [6] and [22]. Q.E.D.

Now assume, using Theorem 1, that we have found the required path s joining q_1 and q_2 in $\mathcal{O}(A')$. The problem of control of the robotic manipulator (or piano movers) then consists of effectively finding a semialgebraic curve $\bar{s}$ in the space A' of control parameters such that the tooling device of the manipulator moves along s when the control parameters move along $\bar{s}$. Mathematically, we are required to find a path $\bar{s} \subset A'$ such that $\mathcal{O}(\bar{s}) = s$. Of course, since the number of degrees of freedom is greater than one, there are many $\bar{s}$'s which "cover" s. However, one has the following result.

Proposition 1: For a given semialgebraic curve s in $\mathcal{O}(A')$, a semialgebraic curve $\bar{s}$ exists in A' such that $\mathcal{O}(\bar{s}) = s$. The diagram $D(\bar{s})$ depends only on $D(s)$, and the equations and inequalities defining $\bar{s}$ can be effectively found.

Proof: Proposition 1 can be derived as a corollary of much more general results [5]–[7], [22] which we will partially state. However, here we would like to show how one may easily construct $\bar{s}$ from s.

Indeed, for each $x \in \mathcal{O}(A')$ set

$$\mathcal{O}^{-1}(x)_{\max} = \max_y \{y \in \mathcal{O}^{-1}(x)\}$$

where the maximum is taken with respect to the lexicographical order on points in $\mathbb{R}^{3n}$. Define

$$\bar{s} = \{\mathcal{O}^{-1}_{\max}(x)\}_{x \in s}.$$

Then it is easy to see $\bar{s}$ is a semialgebraic curve and that $\mathcal{O}(\bar{s}) = s$. Q.E.D.

Remarks:

1) We should note that Proposition 1 does *not* guarantee that the covering curve $\bar{s}$ will be connected (when s is connected), which is certainly a property that one would require physically. However, one may modify the proof of Proposition 1 to obtain the following stronger result. Suppose s connects q_1 and q_2 in $\mathcal{O}(A')$ (see Theorem 1). Then we want to know when a semialgebraic curve $\bar{s} \in A'$ exists such that $\mathcal{O}(\bar{s}) = s$ and such that $\bar{s}$ connects $\bar{q}_1$ and $\bar{q}_2$ for some $\bar{q}_1 \in \mathcal{O}^{-1}(q_1)$ and $\bar{q}_2 \in \mathcal{O}^{-1}(q_2)$. Using the aforementioned techniques (see, e.g., [5]–[7], [22]), one can give an effective procedure for deciding when $\bar{s}$ exists, and when it does exist, for its construction.

2) Proposition 1 is a special case of the following general result.

Theorem 2: For any semialgebraic subset $s \subset \mathcal{O}(A')$, there exists $\bar{s} \subset A'$ semialgebraic such that $\dim \bar{s} = \dim s$, $\mathcal{O}(\bar{s}) = s$, the diagram $D(\bar{s})$ depends only on $D(s)$, and the equations and inequalities defining $\bar{s}$ can be effectively found.

Remark: The existence of $\bar{s}$ can be deduced using basically the same argument as in Proposition 1. However, if we are interested in a detailed description of the structure of $\bar{s}$, we can use an important general result in semialgebraic geometry, namely, the *stratification theorem* (see [5], [7]). Basically, this theorem claims that given an algebraic mapping $f: A_1 \to A_2$, we can effectively stratify A_1 and A_2 into subsets ("strata") such that the restriction of f to each stratum is smooth. Thus the stratification theorem provides a new and effective procedure for dealing with algebraic mappings (e.g., $\mathcal{O}: A' \to \mathbb{R}^3 \times SO(3)$).

Finally, using the metric properties of semialgebraic sets (to be discussed in detail in Sections V and VI), one can derive a number of useful facts about s and $\bar{s}$. Indeed, we conclude this section with the following useful result whose proof may be found in [20] and [22].

Proposition 2: The lengths of the curves s and $\bar{s}$ constructed earlier are bounded by some constant C which may be effectively computed from the diagram $D(A')$.

V. The Space of Bad Positions

In this section we relate the theory of critical values of algebraic mappings to a notion of "bad position" for the robotic manipulator. An obvious problem in the control of the robotic manipulator is that for certain states of the manipulator, large changes in the control parameters may only result in small changes of the tooling device P. For example, consider the situation illustrated in Fig. 3. (Here the angle at the "joint" x_2 is $\pi - \epsilon$; ϵ a "small" positive number.) Clearly, the control parameters (up to first order) only influence the movements of x_{n+1} in directions orthogonal to $\overrightarrow{x_1 x_{n+1}}$. Mathematically, this means that the derivatives of $\mathcal{O}: A' \to \mathbb{R}^3 \times SO(3)$ are small when the manipulator is in such states. This leads us to the following definition.

Definition: 1) Let $A'' \subset A'$ be the set of smooth points of

A' (see [18] for the precise definition). Let $\gamma \in \mathbb{R}$, $\gamma \geqslant 0$. Then we set

$$\Sigma(\gamma) := \{x \in A'' \mid \text{the absolute values of all the minors of maximal size of the Jacobian matrix of } \mathcal{O} \text{ do not exceed } \gamma\}.$$

2) Given $\Sigma(\gamma)$, let

$$\Delta(\gamma) := \mathcal{O}(\Sigma(\gamma)) \subset \mathbb{R}^3 \times SO(3).$$

We call $\Delta(\gamma)$ the *space of γ-bounded positions* of the robotic manipulator.

Clearly, the smaller γ is, the harder it is to control the tooling device P. Thus "small" γ (of course, relative to a given physical requirement) corresponds to the manipulator being in a "bad position." We can now state the following key result whose proof we delay until Section VI (see "Remarks," Section VI).

Theorem 3: Let μ denote the ordinary Lebesque measure on $\mathbb{R}^3 \times SO(3)$. Then

$$\mu(\Delta(\gamma)) \leqslant C\gamma$$

where the constant C depends only on the diagram $D(A')$.

Note then that as $\gamma \to 0$, $\mu(\Delta(\gamma)) \to 0$. Hence Theorem 3 implies the important fact that the space of bad positions (taken relative to γ) has small measure. In the next section we will substantially improve this result.

VI. Metric Entropy

In this section we apply one of the key concepts in topological complexity theory to the robotic manipulator problem, namely, the notion of "entropy." We will see that entropy can be a powerful tool in illuminating many aspects of the digital control of manipulators. We should note that a similar application of this idea has appeared in Zames [25] in connection with feedback sensitivity theory. For other treatments of entropy (and complexity theory), see also the fundamental paper [11] as well as [12], [21], and [22]–[25].

In the preceding sections we attempted to demonstrate the strong relationship between certain problems in robotic control and the geometry of semialgebraic sets (e.g., A' and $\mathcal{O}(A')$). We treated $A' \subset \mathbb{R}^{3n}$ and $\mathcal{O}(A') \subset \mathbb{R}^3 \times SO(3)$ as subsets defined by certain polynomial equations and inequalities. However, in many instances these subsets must be considered pointwise with a prescribed accuracy (this situation occurs also in computer graphics).

From this "pointwise" view point the complexity of a given semialgebraic set A can be described by a certain number $M(\epsilon, A)$ (basically the "ϵ-entropy") which measures the number of points we have to digitally store if we require an accuracy of $\epsilon > 0$ in our computations. We will see that $M(\epsilon, A)$ can be explicitly bounded above in terms of a quantity determined by the diagram $D(A)$ and that $M(\epsilon, A)$ is well behaved under algebraic mappings (e.g., $\mathcal{O}$). Let us now precisely define $M(\epsilon, A)$.

Definition: Let $A \subset \mathbb{R}^n$ be any bounded subset. For any $\epsilon > 0$ we denote by $M(\epsilon, A)$ the minimal number of balls of radius ϵ in $\mathbb{R}^n$ which cover A. $H_\epsilon(A) := \log_2 M(\epsilon, A)$ is called the ϵ *entropy* of A. By abuse of terminology, we will also refer to $M(\epsilon, A)$ as the ϵ entropy of A. (For a detailed treatment of entropy see [11].)

Clearly, to "store" A with accuracy ϵ, it is enough to store all the centers of the $M(\epsilon, A)$ balls of radius ϵ covering A. The behavior of $M(\epsilon, A)$ over various values of ϵ reflects both the "massiveness" of A and its "complexity." We will record here without proof some properties of $M(\epsilon, A)$ (following [22]) which show that for many practical problems the quantity $M(\epsilon, A)$ is much more useful than the Lebesque measure of A.

Properties of $M(\epsilon, A)$: 1) For $A \subset \mathbb{R}^n$,

$$\mu(A) \leqslant K(n) \min_{\epsilon} \epsilon^n M(\epsilon, A)$$

where $K(n)$ is a universal constant only depending on the degrees of freedom n, and $\mu(A)$ denotes the Lebesgue measure of A.

2) $M(\epsilon, A)$ is "stable" with respect to taking open neighborhoods. More precisely, let A_ϵ denote the ϵ neighborhood of A in $\mathbb{R}^n$. Clearly, if a certain number of balls of radius ϵ cover A, then n balls of radius 2ϵ centered at the same points will cover A_ϵ. Hence

$$M(2\epsilon, A_\epsilon) \leqslant M(\epsilon, A).$$

Notice that no such property holds for the Lebesgue measure. For example, $\mu(\mathbb{Q} \cap [0, 1]) = 0$, but any ϵ-neighborhood of $\mathbb{Q} \cap [0, 1]$ covers all of $[0, 1]$.

3) The following property of $M(\epsilon, A)$ is extremely important in the context of computations on a digital computer and hence in the digital control of a robotic manipulator. Namely, if $A \subset \mathbb{R}^n$ has small Lebesgue measure, it can still be very complicated, and in particular, it can be very hard to find even one point *not* in A. For example, for $\mathbb{Q} \subset \mathbb{R}$, $\mu(\mathbb{Q}) = 0$, but yet a digital computer cannot find even one point $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$. From a computational standpoint every number is rational, even though the rational numbers have Lebesgue measure 0.

However, for $M(\epsilon, A)$ we have the following. Let S be any 2ϵ net in $\mathbb{R}^n$ (i.e., $\|x - y\| \geqslant 2\epsilon$ for any $x, y \in S$). Then at most $M(\epsilon, A)$ points in S belong to A. (Indeed, any ball of radius ϵ covers at most one point of S.) Hence if we can compute (or estimate) $M(\epsilon, A)$, we can find "many" points whose prescribed neighborhoods do not intersect A.

The properties 1)–3) apply to *any* bounded subset of $\mathbb{R}^n$. Let us now return to the case (of interest in robotics) when A is a semialgebraic set. Then we have seen that the "complexity" of the semialgebraic representation of A is measured by its diagram $D(A)$. In the following result we bound $M(\epsilon, A)$ (the "pointwise complexity" of A with accuracy ϵ) in terms of $D(A)$.

Theorem 4: Let $A \subset \mathbb{R}^n$ be a k-dimensional semialgebraic set and let $B_r^n \subset \mathbb{R}^n$ be a ball of radius r. Then for any $\epsilon > 0$,

$$M(\epsilon, A \cap B_r^n) \leqslant K_0 + K_1 \left(\frac{r}{\epsilon}\right)^k$$

where K_0, K_1 depend only on $D(A)$.

Proof: For a complete proof see, e.g., [22], [12]. Since it is useful to understand how the proof works, we will consider here only the simplest case of a real algebraic curve $A := \{f = 0\}$ in $\mathbb{R}^2$ where $f(x, y)$ is a polynomial of degree d.

Set $A_r = A \cap B_r^2$. Again for simplicity, we will assume $2r/\epsilon \in \mathbb{N}$. Then clearly we can find a square S_ϵ whose sides have length $2r$ such that $A_r \subset S_\epsilon$. Now divide S_ϵ into $(2r/\epsilon)^2$ subsquares s_i $(i = 1, \cdots, (2r/\epsilon)^2)$ whose sides have length ϵ.

Let t be the number of subsquares which A_r intersects; let u be the number of points of intersection of A_r with the horizontal and vertical lines which define the subsquares s_i; and let v be the number of "ovals" (i.e., compact connected components) of A_r (see [16]) which are properly contained in the s_i $(i = 1, \cdots, (2r/\epsilon)^2)$. Clearly, $u \leqslant 2(2r/\epsilon)d = (4r/\epsilon)d$ since we have $4r/\epsilon$ vertical and horizontal lines and $f(x, y)$ has degree d. Moreover, $v \leqslant (d^2 - d + 2)/2$ by Harnack's theorem. Thus

$$\begin{aligned} M(\epsilon, A_r) &\leqslant t \\ &\leqslant u+v \\ &\leqslant 4d\left(\frac{r}{\epsilon}\right)+\frac{d^2-d+2}{2} \end{aligned}$$

which gives us the required bound. Q.E.D.

Remark: In the control of a robotic manipulator and in particular the piano movers' problem, usually it is essential not only to avoid direct collisions, but also to avoid approaching too closely certain obstacles (e.g., "walls") which can obstruct the motion of the manipulator. As before, this problem can be reduced to the control problem of constraining (if possible) the control parameters from lying in an ϵ neighborhood of a prescribed semialgebraic set. Of course, the choice of a specific ϵ depends on physical considerations, but, certainly, a reasonable choice will involve having the set of admissible parameters sufficiently "large" to be able to control the manipulator. This leads us to the following result.

Corollary 1: Let $A \subset \mathbb{R}^n$, B_r^n be as in Theorem 4. Then if A_ϵ denotes the ϵ neighborhood of A, we have

$$\mu(A_\epsilon \cap B_r^n) \leqslant \bar{K}_0\epsilon^n + \bar{K}_1 r^k \epsilon^{n-k}$$

where $\bar{K}_0$, $\bar{K}_1$ depend only on $D(A)$.

Proof: By properties 1)–3) and Theorem 4, we have

$$\begin{aligned} \mu(A_\epsilon \cap B_r^n) &\leqslant K(n)(2\epsilon)^n M(2\epsilon, A_\epsilon \cap B_r^n) \\ &\leqslant K(n)(2\epsilon)^n M(\epsilon, A \cap B_r^n) \\ &\leqslant K(n)(2\epsilon)^n [K_0 + K_1(r/\epsilon)^k] \end{aligned}$$

from which the corollary immediately follows. Q.E.D.

Remark: Since $\bar{K}_0$ and $\bar{K}_1$ depend only on $D(A)$, Corollary 1 allows us to find an $\epsilon > 0$ such that the measure of the inadmissible control parameters will be uniformly small relative to any obstruction with prescribed "semialgebraic" complexity when $k < n$.

Another natural problem concerning the complexity of control of a robotic manipulator is the following. Suppose we want the tooling device to cover a prescribed semialgebraic subset $\hat{B} \subset \mathbb{O}(A')$ (the space of accessible positions) with a prescribed accuracy ϵ. How complicated a control subspace of parameters $B \subset A'$, such that $\mathbb{O}(B) = \hat{B}$, is needed?

In treating such a problem it is very natural to use ϵ entropy since in many practical situations the prescribed subset $\hat{B}(= \mathbb{O}(B)) \subset \mathbb{R}^3 \times SO(3)$ and the control subspace B will be pointwise stored in a digital computer. Now from Section I the mapping $\mathbb{O}: A' \to \mathbb{R}^3 \times SO(3)$ is a polynomial mapping. Hence the question we have just posed amounts to a problem about the behavior of the ϵ entropy of semialgebraic sets under polynomial mappings. We can, therefore, answer such questions with the following result.

Theorem 5: Let $A \subset \mathbb{R}^n$ be a k-dimensional semialgebraic subset and $B_r^n \subset \mathbb{R}^n$ a ball of radius r. Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $n \geqslant m$, be a polynomial mapping of degree d such that if $J_f(x)$ denotes the Jacobian of f at x, then the operator norm $\|J_f(x)\| \leqslant K$ for all $x \in B_r^n$. Suppose, moreover, that

$$\max_{x\in B_r^n} |m \times m \text{ minors of } J_f(x)| = \gamma.$$

Then

$$M(\epsilon, f(A \cap B_r^n)) \leqslant \hat{K}_0 + \hat{K}_1\gamma\left(\frac{r}{\epsilon}\right)^{k'}$$

where $k' = \min(k, m)$, and the constants $\hat{K}_0$, $\hat{K}_1$ depend only on $D(A)$, K, and d.

Proof: See [22] and "Definition" in Section V.

Remarks:

1) Theorem 5 bounds the entropy of the covered set $f(A)$ in terms of the "semialgebraic complexity" of the control set A. Hence if the set to be covered is given, as well as the accuracy ϵ of the covering, Theorem 5 gives a lower bound for the complexity of the required control.

2) In the special case, when $A = \Sigma(\gamma)$ (see Section V), we return to the problem of "bad positions" considered before. Then Theorem 5 gives the following important bound for the ϵ entropy (and not only the Lebesgue measure) of $\Delta(\gamma)$.

Corollary 2: With the hypotheses of Theorem 5,

$$M(\epsilon, \Delta(\gamma)) \leqslant \hat{\hat{K}}_0 + \hat{\hat{K}}_1\gamma\left(\frac{r}{\epsilon}\right)^m$$

(since $\dim \Sigma(\gamma) = 3n$, and we assume $3n \geqslant m$).

Remarks:

1) For fixed

$$M(\epsilon, \Delta(\gamma)) \to \hat{\hat{K}}_0 \text{ as } \gamma \to 0.$$

This means that the set $\Delta(\gamma)$ is "small" for γ "small" in the sense that in any regular net many points are "far away" from $\Delta(\gamma)$.

2) Let us see why Theorem 3 is a trivial corollary of Corollary 2. Indeed, from Corollary 2 we have

$$(*)\ \epsilon^m M(\epsilon, \Delta(\gamma)) \leqslant K_0\epsilon^m + K_1\gamma r^m.$$

However, from Section VI-B, property 1),

$$\mu(\Delta(\gamma)) \leqslant K(n)\epsilon^m M(\epsilon, \Delta(\gamma)).$$

Therefore, taking $\epsilon \to 0$ on both sides of (*) immediately gives Theorem 3.

VII. Conclusion

In this paper we attempted to show that many problems connected with robotic manipulators amount to certain questions about the geometry of real semialgebraic sets. In this sense, our work is a continuation of work such as [14] and [15].

However, given the powerful techniques available today in real semialgebraic geometry, e.g., the stratification and triangulation theorems of Hardt [5], [7] and Hironaka [9], the results concerning the notion of ϵ entropy, and, in general, the metric properties of semialgebraic sets and mappings (see, e.g., [6], [20], [22]–[24]), we have tried to point out the relevance of more sophisticated results from this theory to practical problems in robotics. In this sense, our paper may be regarded as a guide to the available results in semialgebraic geometry for those engineers and mathematicians working in robotics.

Finally, since both the fields of robotics and semialgebraic geometry are progressing rapidly, it seems that given their close connection in certain problem areas both fields have much to gain from and contribute to each other. This paper may be regarded as a first attempt in this direction.

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